9 Appendix: Proofs

Proof of Theorem 1. i. According to Lemma 1[4] in Zhao and Atkins (2007), we need to show $\pi_i(p_i) =^{def} \pi_i^d - (w_i - \beta_i)(k - mp_i) + (p_i - \beta_i)E[\min\{D_i^s, k - mp_i\}]$ to be quasiconcave in p_i.

$\frac{d\pi_i(p_i)}{dp_i} = \frac{d\pi_i^d}{dp_i} + (w_i - \beta_i)m + E[\min\{D_i^s, k - mp_i\}] - m(p_i - \beta_i)\Pr(D_i^s > k - mp_i)$

$\frac{d^2\pi_i(p_i)}{dp_i^2} = \frac{d^2\pi_i^d}{dp_i^2} + \Pr(D_i^s > k - mp_i)[-2m - m^2(p_i - \beta_i)r_{D_i^s}(k - mp_i)]$

If $m \geq 0$, then $\pi_i(p_i)$ is strictly concave in p_i, done.

If $m < 0$, then let $n = -m > 0$. According to (A), $d^2\pi_i^d/dp_i^2 < 0$ and is decreasing in p_i. If $[2 - n(p_i - \beta_i)r_{D_i^s}(k + np_i)] < 0$, then $\pi_i(p_i)$ is strictly concave in p_i, done. Otherwise, by (B), $[2 - n(p_i - \beta_i)r_{D_i^s}(k + np_i)]$ decreases as p_i increases from w_i to $p_i^{\max}$. Hence $d^2\pi_i(p_i)/dp_i^2$ either changes sign at most once from positive to negative or is always negative. Thus, whenever $d\pi_i(p_i)/dp_i$ turns negative, it remains negative, and $\pi_i(p_i)$ is quasiconcave in p_i. So, function (1) is quasiconcave in (p_i, y_i) and a pure-strategy Nash equilibrium exists.

ii. We first show that maxima of function (1) are interior, then that equations (2)-(3) have a unique solution.

Note that $\lim_{p_i \to p_i^{\max}} d\pi_i/dp_i < 0, \lim_{y_i \to y_i^{\max}} d\pi_i/dy_i = -(w_i - \beta_i) < 0, \lim_{p_i \to w_i} d\pi_i/dp_i > 0$, and $\lim_{y_i \to 0} d\pi_i/dy_i = p_i - w_i > 0$. So boundary solutions are not optimal. Next we show that a unique maximizer solves (2)-(3), satisfying $Q(p_i) =^{def} \partial^2\pi_i^d/\partial p_i^2 + \Pr(D_i^s > y_i)/[(p_i - \beta_i)r_{D_i^s}(y_i)] < 0$.

[4]Proved in Zhao and Atkins (2007), a bivariate function $g(x_1, x_2)$ is jointly quasiconcave in two variables iff every "vertical slice" of the function is quasiconcave, or more formally, iff $g(x_1, x_2)$ is quasiconcave given $mx_1 + x_2 = k$ for any real values m and k.

Uniquely solve $y_i(p_i)$ from (3) and substitute into (2), resulting in

$$\partial\pi_i^d/\partial p_i + E[\min\{D_i^s, y_i(p_i)\}] = 0 \tag{A1}$$

Define $J(p_i) =^{def} d\pi_i^d/dp_i + E[\min\{D_i^s, y_i(p_i)\}]$, where $J(w_i) > 0$ and $J(p_i^{\max}) < 0$, and $dJ(p_i)/dp_i = Q(p_i)$.

Note that the last term of $Q(p_i)$ decreases with p_i and approaches zero by (B). Also, if (A) holds, then $d^2J(p_i)/dp_i^2 < 0$, so $Q(p_i)$ decreases with p_i and approaches $\partial^2\pi_i^d/\partial p_i^2$ as p_i goes to $p_i^{\max}$. Thus $J(p_i)$ is strictly concave, starts positive, and finally strictly decreases to negative. So there is a unique solution for equation (A1), at $Q(p_i) < 0$.

Proof of Proposition 1. Redefine retailer j's strategy space as $\widetilde{y}_j = -y_j$ and $\widetilde{p}_j = -p_j$. It can be shown that $\partial^2\pi_i/\partial p_i\partial y_i \geq 0, \partial^2\pi_i/\partial p_i\partial\widetilde{p}_j = 0$, $\partial^2\pi_i/\partial p_i\partial\widetilde{y}_j = 0$, $\partial^2\pi_i/\partial y_i\partial\widetilde{y}_j \geq 0$ and $\partial^2\pi_i/\partial y_i\partial\widetilde{p}_j = 0$. So π_i is supermodular in (p_i, y_i) and has increasing difference in $(p_i, \widetilde{p}_j)$, $(p_i, \widetilde{y}_j)$, $(y_i, \widetilde{p}_j)$ and $(y_i, \widetilde{y}_j)$. Similarly we show the supermodularity and increasing difference of π_j. According to Milgrom and Roberts (1990), Theorem 4, the game is supermodular, and a pure Nash equilibrium exists (Topkis 1998, Theorem 4.2.1).

Proof of Theorem 2. A sufficient condition (Contraction Mapping Theorem 3.4, Friedman 1990) requires $|\frac{\partial^2\pi_i}{\partial p_i^2}| > \sum_{j\neq i}(|\frac{\partial^2\pi_i}{\partial p_i\partial p_j}| + |\frac{\partial^2\pi_i}{\partial p_i\partial y_j}|) + |\frac{\partial^2\pi_i}{\partial p_i\partial y_i}|$ and $|\frac{\partial^2\pi_i}{\partial y_i^2}| > \sum_{j\neq i}(|\frac{\partial^2\pi_i}{\partial y_i\partial p_j}| + |\frac{\partial^2\pi_i}{\partial y_i\partial y_j}|) + |\frac{\partial^2\pi_i}{\partial y_i\partial p_i}|$ for uniqueness, which are

$-\frac{\partial^2\pi_i^d}{\partial p_i^2} > \sum_{j\neq i}|\frac{\partial^2\pi_i^d}{\partial p_i\partial p_j}| + \Pr(D_i^s > y_i) + \sum_{j\neq i}\gamma_{ji}\Pr(D_i^s < y_i, \epsilon_j > y_j)$,

$1 > 1/[(p_i - \beta_i)r_{D_i^s}] + \sum_{j\neq i}\gamma_{ji}f_{D_i^s|\epsilon_j > y_j}(y_i)\Pr(\epsilon_j > y_j)/f_{D_i^s}(y_i)$

The required results is obtained by

(i) $1/[(p_i - \beta_i)r_{D_i^s}] \leq 1/[(w_i - \beta_i)r_{D_i^s}]$,

(ii) $f_{D_i^s|\epsilon_j>y_j}(y_i)\Pr(\epsilon_j > y_j)/f_{D_i^s}(y_i) < 1$, and

(iii) $\Pr(D_i^s > y_i) + \sum_{j\neq i}\gamma_{ji}\Pr(D_i^s < y_i, \epsilon_j > y_j) \leq \max\{1, \sum_{j\neq i}\gamma_{ji}\}$.

Proof of Proposition 2. An immediate result from Theorem 1 is that there exists a symmetric equilibrium for the game (Cachon and Netessine 2004). Now we show that given $p_i = p_{-i} = p$ and $y_i = y_{-i} = y$ and a symmetric demand and cost function, there exists a unique symmetric equilibrium. That is, the solution from (2) and (3) under symmetry,

$$-(w-\beta) + (p-\beta)\Pr(D_i^s \geq y) = 0 \text{ and} \tag{A2}$$

$$\partial^2\pi_i^d/\partial p_i^2 + E[\min\{D_i^s, y\}] = 0, \tag{A3}$$

is unique. Define $J(p) =^{def} \partial^2\pi_i^d/\partial p_i^2 + E\min\{D_i^s, y(p)\}$, where $y(p)$ is the unique solution of equation $(A2)$. Now $J(w) > 0$, and $J(p^{\max}) < 0$. Also $dJ(p)/dp = \partial^2\pi_i^d/\partial p_i^2 + \sum_{j\neq i}\partial^2\pi_i^d/\partial p_i\partial p_j + A(y)y'(p)$, where $A(y) =^{def} \partial E[\min\{D_i^s, y\}]/\partial y$ and $y'(p) =^{def} dy(p)/dp$.

First, we show that $A(y) > 0$ and decreases in y. Then we show that $y'(p) > 0$ and decreases in y. Then $dJ(p)/dp$ can be either always negative, or start positive but decrease to negative and stay negative. Then there is a unique p that solves $J(p) = 0$, at $dJ(p)/dp < 0$. Thus a unique symmetric equilibrium exists.

Using a methodology introduced by Netessine and Rudi (2003) for differentiation, we have $A(y) = \partial E[\min\{D_i^s, y\}]/\partial y = \Pr(D_i^s > y_i) - (N-1)\gamma\Pr(D_i^s < y_i, \epsilon_j > y_j)$

$\geq \Pr(D_i^s > y_i) - \Pr(D_i^s < y_i, \epsilon_j > y_j) \geq \Pr(D_i^s > y_i) - \Pr(\epsilon_i > y_i) \geq 0$.

Also note that $E[\min\{D_i^s, y\}] = E[\min\{\epsilon_i, y\}] + E\min[\{(y-\epsilon_i)^+, (N-1)\gamma(\epsilon_j - y)^+\}]$. So

$A(y) = \partial(E[\min\{\epsilon_i, y\}] + E\min[\{(y-\epsilon_i)^+, (N-1)\gamma(\epsilon_j - y)^+\}])/\partial y$

$= \Pr(\epsilon_i > y) + \Pr(y - (N-1)\gamma(\epsilon_j - y) < \epsilon_i < y) - (N-1)\gamma\Pr(y < \epsilon_j < y + (y-\epsilon_i)/(N -$

$1)\gamma)$, which decreases with y.

From $(A2)$, $y'(p) = \Pr(D_i^s > y)/(p-\beta)(\partial \Pr(D_i^s > y)/\partial y) = 1/(p-\beta)r_{D_i^s}(y)$, which decreases in y under the IFR assumption for D_i^s and the fact that D_i^s stochastically decreases with y.

Proof of Theorem 3. Given (p_{-i}^c, y_{-i}^c), the unique best response of retailer i will be (p_i^c, y_i^c) if functions (2)-(3) are equivalent to (4)-(5). Thus, $w_i^* = c_i - \sum_{j\neq i}(p_j^c - c_j)L_j^{(i)}(\overrightarrow{p^c})/L_i^{(i)}(\overrightarrow{p^c})$ and $\beta_i^* = [w_i^* - p_i^c \Pr(D_i^s > y_i^c)]/\Pr(D_i^s < y_i^c) = p_i^c - (p_i^c - w_i^*)/\Pr(D_i^s < y_i^c)$. This approach has been justified by Winter (1993), Cachon (1999), and Tsay and Agrawal (2000). It can be shown that $c_i < w_i^* < p_i^c$, and $\beta_i^* = p_i^c + (L_i(\overrightarrow{p^c}) + E[\min(D_i^s, y_i^c))/[L_i^{(i)}(p^c)\Pr(D_i^s < y_i^c)] < w_i^*$.

Next, we prove that $(\overrightarrow{p_i^c}, \overrightarrow{y_i^c})$ is a Pareto-dominant equilibrium for the whole game.

Assume there is another equilibrium $(\overrightarrow{p_i^o}, \overrightarrow{y_i^o})$ that Pareto-dominates $(\overrightarrow{p_i^c}, \overrightarrow{y_i^c})$. Then at $(\overrightarrow{p_i^o}, \overrightarrow{y_i^o})$, at least one player gets better off without making any other player worse off than at $(\overrightarrow{p_i^c}, \overrightarrow{y_i^c})$. But this is not possible since at $(\overrightarrow{p_i^c}, \overrightarrow{y_i^c})$, the total supply chain's profit is no less than that at $(\overrightarrow{p_i^o}, \overrightarrow{y_i^o})$. If one player is better off at $(\overrightarrow{p_i^o}, \overrightarrow{y_i^o})$, there must be at least one player getting worse off at $(\overrightarrow{p_i^o}, \overrightarrow{y_i^o})$. So $(\overrightarrow{p_i^c}, \overrightarrow{y_i^c})$ is a Pareto-dominant equilibrium.

Assume that the optimum for the system is the unique. If the payoffs are transferrable among players, then similar reasoning shows that it is the unique Pareto-dominant equilibrium.

Proof of Proposition 3. With price competition only, $\beta_i^* = p_i^c - (p_i^c - w_i^*)/\Pr(D_i^s < y_i^c) = (-c_i + w_i^*)/\Pr(D_i^s < y_i^c) > 0$. With inventory competition only, $w_i^* = c_i$ and $\beta_i^* = -\sum_{j\neq i} p_j^c \gamma_{ij} \Pr(D_j^s < y_j^c, \epsilon_i > y_i^c)/\Pr(D_j^s < y_j^c) < 0$.

Proof of Proposition 4.

(i) To simplify the presentation, let $H =^{def} E[\min\{D_i^s, y_i\}]$. Then $\partial H/\partial y_i = \Pr(D_i^s > y_i)$

and $\partial H/\partial y_j = -\gamma \Pr(D_i^s < y_i, \epsilon_j > y_j)$. We first show that at a symmetric equilibrium (solution of $(A2)$ and $(A3)$), we have

$$\frac{\partial^2 \pi_i^d}{\partial p_i^2} + \sum_{j\neq i} \frac{\partial^2 \pi_i^d}{\partial p_i \partial p_j} - \left(\frac{\partial H}{\partial y_i} + \sum_{j\neq i} \frac{\partial H}{\partial y_j}\right) \frac{\partial H/\partial y_i}{(p-\beta)(\partial^2 H/\partial y_i^2 + \sum_{j\neq i} \partial^2 H/\partial y_i \partial y_j)} < 0.$$

Following Theorem 1, the symmetric equilibrium price is solved by equation $(A1)$. That is, $J(p) = \partial \pi_i^d/\partial p_i + E[\min\{D_i^s, y(p)\}] = 0$, where $y(p)$ is the solution to equation (3) after setting $y_i = y$ for all i. As in part ii of the proof of Theorem 1, the solution p to $J(p) = 0$ must occur when $dJ(p)/dp < 0$. Note that

$$\frac{dJ(p)}{dp} = \frac{\partial^2 \pi_i^d}{\partial p_i^2} + \sum_{j\neq i} \frac{\partial^2 \pi_i^d}{\partial p_i \partial p_j} + \left(\frac{\partial H}{\partial y_i} + \sum_{j\neq i} \frac{\partial H}{\partial y_j}\right) \frac{dy}{dp}$$

where $\frac{dy}{dp} = -\frac{\partial H/\partial y_i}{(p-\beta)(\partial^2 H/\partial y_i^2 + \sum_{j\neq i} \partial^2 H/\partial y_i \partial y_j)}$ is derived from equation (3). Hence this intermediate result.

The main result can now be derived. Differentiating (2) and (3) with respect to β, we have

$$\left(\frac{\partial^2 \pi_i^d}{\partial p_i^2} + \sum_{j\neq i} \frac{\partial^2 \pi_i^d}{\partial p_i \partial p_j}\right) \frac{dp^*}{d\beta} + \left(\frac{\partial H}{\partial y_i} + \sum_{j\neq i} \frac{\partial H}{\partial y_j}\right) \frac{dy^*}{d\beta} = 0 \text{ and}$$

$$\frac{\partial H}{\partial y_i} \frac{dp^*}{d\beta} + (p-\beta)\left(\frac{\partial^2 H}{\partial y_i^2} + \sum_{j\neq i} \frac{\partial^2 H}{\partial y_i \partial y_j}\right) \frac{dy^*}{d\beta} = -\left(1 - \frac{\partial H}{\partial y_i}\right).$$

Using Cramer's rule, we have

$$\frac{dp^*}{d\beta} = \begin{vmatrix} 0 & \left(\frac{\partial H}{\partial y_i} + \sum_{j\neq i} \frac{\partial H}{\partial y_j}\right) \\ -\left(1 - \frac{\partial H}{\partial y_i}\right) & (p-\beta)\left(\frac{\partial^2 H}{\partial y_i^2} + \sum_{j\neq i} \frac{\partial^2 H}{\partial y_i \partial y_j}\right) \end{vmatrix} \Big/ \begin{vmatrix} \left(\frac{\partial^2 \pi_i^d}{\partial p_i^2} + \sum_{j\neq i} \frac{\partial^2 \pi_i^d}{\partial p_i \partial p_j}\right) & \left(\frac{\partial H}{\partial y_i} + \sum_{j\neq i} \frac{\partial H}{\partial y_j}\right) \\ \frac{\partial H}{\partial y_i} & (p-\beta)\left(\frac{\partial^2 H}{\partial y_i^2} + \sum_{j\neq i} \frac{\partial^2 H}{\partial y_i \partial y_j}\right) \end{vmatrix},$$

$$\frac{dy^*}{d\beta} = \begin{vmatrix} \left(\frac{\partial^2 \pi_i^d}{\partial p_i^2} + \sum_{j\neq i} \frac{\partial^2 \pi_i^d}{\partial p_i \partial p_j}\right) & 0 \\ \frac{\partial H}{\partial y_i} & -\left(1 - \frac{\partial H}{\partial y_i}\right) \end{vmatrix} \Big/ \begin{vmatrix} \left(\frac{\partial^2 \pi_i^d}{\partial p_i^2} + \sum_{j\neq i} \frac{\partial^2 \pi_i^d}{\partial p_i \partial p_j}\right) & \left(\frac{\partial H}{\partial y_i} + \sum_{j\neq i} \frac{\partial H}{\partial y_j}\right) \\ \frac{\partial H}{\partial y_i} & (p-\beta)\left(\frac{\partial^2 H}{\partial y_i^2} + \sum_{j\neq i} \frac{\partial^2 H}{\partial y_i \partial y_j}\right) \end{vmatrix}.$$

Note that $\frac{\partial^2 \pi_i^d}{\partial p_i^2} + \sum_{j\neq i} \frac{\partial^2 \pi_i^d}{\partial p_i \partial p_j} < 0$ (Vives 1999), $\frac{\partial H}{\partial y_i} > 0$, $1 - \frac{\partial H}{\partial y_i} > 0$, $\frac{\partial^2 H}{\partial y_i^2} < 0$, $\frac{\partial H}{\partial y_j} < 0$, $\frac{\partial^2 H}{\partial y_i \partial y_j} < 0$, in addition, $\frac{\partial H}{\partial y_i} + \sum_{j\neq i} \frac{\partial H}{\partial y_j} > 0$. Then $dp^*/d\beta > 0$ and $dy^*/d\beta > 0$.

(ii) Differentiating (2) and (3) with respect to w, we have

$(\frac{\partial^2 \pi_i^d}{\partial p_i^2} + \sum_{j \neq i} \frac{\partial^2 \pi_i^d}{\partial p_i \partial p_j})\frac{dp^*}{dw} + (\frac{\partial H}{\partial y_i} + \sum_{j \neq i} \frac{\partial H}{\partial y_j})\frac{dy^*}{dw} = -\frac{\partial^2 \pi_i^d}{\partial p_i \partial w}$ and

$\frac{\partial H}{\partial y_i}\frac{dp^*}{dw} + (p - \beta)(\frac{\partial^2 H}{\partial y_i^2} + \sum_{j \neq i} \frac{\partial^2 H}{\partial y_i \partial y_j})\frac{dy^*}{dw} = 1.$

Note that $\frac{\partial^2 \pi_i^d}{\partial p_i \partial w} > 0$. It can be shown that the only combination that cannot hold is $dp^*/dw < 0$ and $dy^*/dw > 0$.

Proof of Proposition 5. With linear demand, $w_i^* = c_i + \sum_{j \neq i}(p_j^c - c_j)\theta/(b + \theta)$, since p_j^c is unaffected by θ, $dw_i^*/d\theta > 0$. By equation (9), $d\beta_i^*/d\theta > 0$.

Proof of Proposition 6. Substituting (w_i^*, β_i^*) and $(\overrightarrow{p^c}, \overrightarrow{y^c})$ into function (1), we have

$\pi_i = (p_i^c - w_i^*)[L_i(\overrightarrow{p^c}) - y_i^c \Pr(\epsilon_i > y_i^c)/\Pr(\epsilon_i < y_i^c) + E[\min\{\epsilon_i, y_i^c\}]/\Pr(\epsilon_i < y_i^c)]$.

Notice that $\pi_i^c = (p_i^c - c_i)[L_i(\overrightarrow{p^c}) - y_i^c c_i/(p_i^c - c_i) + p_i^c E[\min\{\epsilon_i, y_i^c\}]/(p_i^c - c_i)]$.

By equations (4)-(5), we have

$c_i/(p_i^c - c_i) = \Pr(\epsilon_i > y_i^c)/\Pr(\epsilon_i < y_i^c)$ and

$p_i^c/(p_i^c - c_i) = 1/\Pr(\epsilon_i < y_i^c)$.

Then $\pi_i/\pi_i^c = (p_i^c - w_i^*)/(p_i^c - c_i) = [p_i^c - c_i + \sum_{j \neq i}(p_j^c - c_j)L_j^{(i)}(\overrightarrow{p^c})/L_i^{(i)}(\overrightarrow{p^c})]/(p_i^c - c_i)$

$= 1 + \sum_{j \neq i} L_j^{(i)}(\overrightarrow{p^c})/L_i^{(i)}(\overrightarrow{p^c}) = 1 - (n-1)\theta/(b + \theta)$.

The second equality holds because $p_i^c - c_i = p_j^c - c_j$ in a symmetric game, and the last equality holds for the linear demand function.