Contracting under vendor managed inventory systems using holding cost subsidies

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Appendix: Proofs of all results

Observation 2.1:

We first show that the two constraints in **(P)** are satisfied if and only if $A_r < A_m$.

In the VMI system, when the retailer announces a rent ρ, the manufacturer picks the order quantity $Q^V = \sqrt{2A_m D/\rho}$ and his total cost is $J_M^V = \sqrt{2A_m D\rho}$. The corresponding cost of the retailer is $J_R^V = S(\rho)$, where $S(\rho) = \frac{\sqrt{\rho}(A_r - A_m)\sqrt{D}}{\sqrt{2A_m}} + \frac{H\sqrt{A_m D}}{\sqrt{2\rho}}$. On the other hand, in the RMI system, the retailer's optimal order quantity is $Q^R = \sqrt{2A_r D/H}$ and the corresponding costs of the retailer and the manufacturer are $J_R^R = \sqrt{2A_r DH}$ and $J_M^R = A_m\sqrt{\frac{HD}{2A_r}}$, respectively. We need to find a rent ρ such that $J_M^V \le J_M^R$ and $J_R^V \le J_R^R$. Note that the former can be true if and only if $\rho \le \frac{A_m H}{4A_r} = \rho^*$. Now, if $A_r < A_m$,

$$S(\rho^*) = (A_r - A_m)\sqrt{\frac{DH}{8A_r}} + \sqrt{2A_r DH} \le J_R^R,$$

Thus, if $A_r < A_m$, we have shown that there exists a rent parameter ρ^* that ensures that the VMI system is better for both players.

To finish the proof, let $A_r > A_m$. $S(\rho)$ has a unique minimum at $\tilde{\rho} = \frac{HA_m}{A_r - A_m}$ decreasing when $\tilde{\rho} \ge \rho$ and increasing when $\tilde{\rho} < \rho$. We finish the proof by noting that $S(\rho^*) > J_R^R$ when $A_r < A_m$, and $\tilde{\rho} \ge \rho*$.

To solve (P), we thus focus on the case when $A_r < A_m$. In this case, note that $S(\rho)$ is a decreasing function. Thus (P) is solved at $\rho = \rho^*$. To show that this does not coordinate the channel, we observe that $\rho^\bullet = \dfrac{A_m H}{A_r + A_m}$ is the unique rent that coordinates the channel.

Lemma 3.1: $n(r) = \int_r^\infty (\xi - r) f(\xi) d\xi = \int_r^\infty (\xi - r)\beta e^{-\alpha\xi} d\xi$

Integrating by parts and canceling common terms, we get:

$$n(r) = \beta \int_r^\infty \xi e^{-\alpha\xi} \; d\xi - \beta \int_r^\infty r e^{-\alpha\xi} d\xi = \frac{\beta e^{-\alpha\xi}}{\alpha^2}$$

$$\overline{F}(r) = \int_r^\infty f(\xi) d\xi = \int_r^\infty \beta e^{-\alpha\xi} d\xi = \frac{\beta e^{-\alpha r}}{\alpha}$$

So, $n(r) = \overline{F}(r)/\alpha$.

Lemma 3.2: Since the cost functions $J_{SC}^C(Q,r)$, $J_M^V(Q,r)$, and $J_R^R(Q,r)$ are quite similar, we will look at a general cost function of the following form:

$$J(Q,r) = AD/Q + (\theta Q + \rho r) + [\int_r^\infty f(x) dx] PD/Q$$

Since the sum of convex functions is convex, to show that $J(Q,r)$ is jointly convex, it suffices to show that $b(Q,r) = [\int_r^\infty f(x) dx] PD/Q$ is jointly convex in (Q,r).

Let $H(Q, r)$ be the Hessian matrix of $b(Q, r)$. To show convexity, we need to show positive semi-definiteness of H. Since we have a 2x2 matrix, it is enough to show that the determinant and the diagonal entries are positive. From the above expressions it is clear that the latter is true. Now, WLOG, set $PD = 1$. The lead time demand is $f(x) = \beta e^{-\alpha x}$, $\alpha \geq 0$ and for ease of exposition and WLOG, ignore the constant term β. We then have the following:

$$\frac{d^2b}{dQ^2} = \frac{2}{Q^3}\int_r^\infty f(x)dx; \quad \frac{d^2b}{dr^2} = \frac{-1}{Q} f'(r); \quad \frac{d^2b}{dQdr} = \frac{1}{Q^2} f(r);$$

The diagonal terms of this Hessian $H(Q, r)$ are again positive. Also, note that

$$Det(H(Q,r)) \geq 0 \Leftrightarrow -2f'(r)(1-F(r)) \geq [f(r)]^2 \Leftrightarrow 2e^{2\alpha r} > e^{2\alpha r}.$$

This implies convexity.

Observation 3.1: Using equations (8) and (9) in section 3, we have

$$Q^V = \frac{1}{\alpha}\left[1+\sqrt{1+2DA_m\alpha^2/\rho}\right] \text{ and } r^V = \frac{1}{\alpha}\left[\ln\left(\frac{P_m D\alpha/\rho}{1+\sqrt{1+2DA_m\alpha^2/\rho}}\right)\right];$$

It is immediate that in general, there is no ρ such that $Q^V = Q^C$ and $r^V = r^C$ simultaneously.

Theorem 3.1: (1) We will start with the case where a penalty is incurred for every stockout occasion. We need to show that there exists a rent ρ such that $J^V_{SC}(Q^V, r^V) \leq J^R_{SC}(Q^R, r^R)$**.** We will demonstrate this by taking two cases.

Case 1: Let $A_m/P_m \geq A_r/P_r$. Consider a rent ρ such that $r^V = r^R = r$. From (5) and (8), we have

$$Q^V = \frac{Q^R HP_m}{\rho P_r}$$

This implies that $Q^V \geq Q^R$ since $A_m/P_m \geq A_r/P_r$. We then have:

$$\begin{aligned} J^V_{SC}(Q^V, r) &= [(A_r + A_m) + (P_m + P_r)\bar{F}(r)]D/Q^V + HQ^V/2 - H\mu \\ &= [(A_r + A_m) + (P_m + P_r)\bar{F}(r)]D/Q^V + DH[A_m + P_m\bar{F}(r)]/\rho Q^V \end{aligned}$$

$$\begin{aligned} J^R_{SC}(Q^R, r) &= [(A_r + A_m) + (P_m + P_r)\bar{F}(r)]D/Q^R + HQ^R/2 - H\mu \\ &= [(A_r + A_m) + (P_m + P_r)\bar{F}(r)]\frac{DHP_m}{Q^V\rho P_r} + DP_r[A_m + P_m\bar{F}(r)]/P_m Q^V \end{aligned}$$

Taking the difference and canceling common terms, we have,

$$J_{SC}^{V}(Q^{V},r) - J_{SC}^{R}(Q^{R},r) = \frac{D}{Q^{V}}\{(1-\frac{HP_m}{\rho P_r})[(A_r + A_m) + (P_m + P_r)\bar{F}(r) - (A_m P_r / P_m + \bar{F}(r)P_m)]\}$$

Rearranging terms and canceling common ones, we have,

$$J_{SC}^{V}(Q^{V},r) - J_{SC}^{R}(Q^{R},r) = \frac{D}{Q^{V}}\{(1-\frac{HP_m}{\rho P_r})[A_r + A_m(1-P_r/P_m) + P_r\bar{F}(r)]\}$$

Note that the expression in [.] is positive because $P_r \leq P_m$. Since our choice of rent implies that

$Q^{V} \geq Q^{R} \Rightarrow \frac{HP_m}{P_r \rho} \geq 1$, we are done.

Case 2: Let $A_m/P_m < A_r/P_r$.

Consider a rent ρ such that $Q^V = Q^R = Q$. As before, taking the difference and canceling common terms, we have:

$$\begin{aligned}
J_{SC}^{V}(Q,r^{V}) - J_{SC}^{R}(Q,r^{R}) &= \frac{D}{Q}(P_m + P_r)\bar{F}(r^{V}) - \frac{D}{Q}(P_m + P_r)\bar{F}(r^{R}) - H(r^{V} - r^{R}) \\
&= \frac{H}{\alpha}[\ln(\frac{P_m D}{\rho Q}) - \ln(\frac{P_r D}{HQ})] + \frac{P_r + P_m}{\alpha}[\frac{\rho}{P_m} - \frac{H}{P_r}] \\
&= \frac{H}{\alpha}\{\ln(\frac{P_m A_r}{P_r A_m}) + \frac{A_m P_r}{A_r P_m} + \frac{A_m}{A_r} - \frac{P_m}{P_r} - 1\}
\end{aligned}$$

Set $P_m/P_r = y$ and $A_m/A_r = x$. Let $G(x) = \ln(\frac{y}{x}) + \frac{x}{y} + x - y - 1$. We then have

$$J_{SC}^{V}(Q,r^{V}) - J_{SC}^{R}(Q,r^{R}) = \frac{H}{\alpha}G(x).$$

$G(x)$ is a convex function and has a unique minimum at $x = y/(y+1) < 1$. Further, $G(y) = 0$ and $P_m/P_r = y \geq 1$. $G(x)$ has another zero at some $x^* < y$. Hence, it is easy to see that $G(x) \leq 0$ when $x \in [x^*, y]$ and is positive everywhere else. By our assumption $x \leq y$. To finish the proof, we need to ensure that $x \geq x^*$. As y increases, x^* decreases. When $y = 1$, we get $x^* = 0.203$ and this corresponds to $A_r = 4.9A_m$. Hence, if $A_r \leq 5A_m$, we can ensure that $G(x) \leq 0$.

(2) Now consider the case where a penalty cost for a stockout is incurred for every unit stocked out. In this case, the total cost of the supply chain is:

$$J_{SC}^{C}(Q,r) = (A_r + A_m)D/Q + H(Q/2 + r - \mu) + (P_m + P_r)n(r)D/Q \ .$$

The only difference in the cost function is that we have *n(r)* instead of $\bar{F}(r)$. From Lemma 1, $n(r) = \bar{F}(r)/\alpha$. Also, $\bar{F}(r) = \beta e^{-\alpha r}/\alpha = f(r)/\alpha$. So, it can be shown easily that the only difference in the proof for this case, relative to the earlier one, is that we replace $\bar{F}(r)$ with *n(r)* and *f(r)* with $\bar{F}(r)$. □

Theorem 4.1:

We analyze the case when the production rate is greater than *D* but finite. As mentioned in the discussion following Theorem 4.1, the case of infinite production simply yields *X=0* in the discussion below.

To prove **(1)** we need only produce a pair (θ, ρ) that results in channel coordination. For a given pair (θ, ρ), the manufacturer's optimal response *(Q, r)*,when $\mu > D$ satisfies the following pair of equations:

$$\begin{aligned} &\frac{-A_m D}{Q^2} - \frac{P_m D}{Q^2}\bar{F}(r) + \theta + X/2 = 0 \\ &\frac{-P_m D}{Q} f(r) + \rho = 0 \end{aligned} \qquad \text{(A1)}$$

Let the centralized system or first best solution be (Q^C, r^C). It is clear that there is a unique pair (θ^*, ρ^*) that satisfies the pair of equations (A1), when we set (Q, r) = (Q^C, r^C). But we do not claim that this solution can allocate fractions of the savings such that the participants do better than in a RMI system. However, combined with side payments, the contract can allocate the first best profit arbitrarily to the players.

To prove **(2),** we produce a $(\theta, \rho) \geq 0$ such that both players are better in the VMI system than in the RMI system. We will represent the manufacturer's optimal response in the VMI system by (Q^V, r^V), which satisfies equation (A1) above.

We will denote by (Q^R, r^R) the decisions made by the retailer in an RMI system. The player's costs in the VMI and RMI systems are denoted, as before, by J_M^V, J_R^V, J_R^R, and J_M^R. We need to show that there exists a contract S_1 or equivalently a pair (θ, ρ) such that $J_M^V - J_M^R \leq 0$ and $J_R^V - J_R^R \leq 0$. First choose $(\theta, \rho) \geq 0$ such that $Q^V = Q^R$. Note that the expression for Q^R is the same as when the production rate was infinite.

Using $\left(Q^V\right)^2 = \dfrac{2D[A_m + P_m \bar{F}(r_V)]}{\theta + X/2}$ and $\rho = \dfrac{P_m \bar{F}(r_V)}{(\theta + X/2)Q^V}$, we get

$$Q^V = \frac{\left(\rho + \sqrt{\rho^2 / P_m^2 + 8DA_m\theta}\right)}{2(\theta + X/2)}.$$

So, for any $\rho > 0$, there exists $\theta > 0$ such that $Q^V = Q^R = Q$. Now, let

$$Y = J_M^V - J_M^R = \frac{P_m D}{Q}[\bar{F}(r^V) - \bar{F}(r^R)] + (\theta + X/2)Q + \rho r^V$$

$$Z = J_R^V - J_R^R = \frac{P_r D}{Q}[\bar{F}(r^V) - \bar{F}(r^R)] - (\theta Q + (\rho - H)r^V) - Hr^R$$

Using $\bar{F}(r) = f(r)/\alpha$, we have $Q^V = Q^R \Rightarrow \bar{F}(r^R) = \dfrac{P_m H}{P_r \rho}\bar{F}(r^V)$ and writing Y as a function of ρ:

$$Y = Y(\rho) = \frac{\bar{F}(r^R)D}{Q}\left(\frac{P_r \rho}{H} - P_m\right) + (\theta + X/2)Q^V + \rho r^R - \frac{\rho}{t}\ln\left[\frac{\rho P_r}{P_m H}\right]$$

But $Y(\infty) \to -\infty$ and $Y(\frac{HP_m}{P_r}) > 0$, and Y continuous for $\rho > 0$ implies that there exists a $\rho^* > H$ such that $Y(\rho^*) < 0$. But notice that if $Y \le 0$ and $\theta, \rho > 0$ then $\bar{F}(r^V) - \bar{F}(r^R) \le 0$. Thus when $\rho = \rho^*$, we have $Z \le 0$. This finishes the proof. □

Lemma 5.1: Part (1) of the lemma is straightforward, so we will prove part (2) of the lemma., Without any loss of generality and for ease of exposition, we ignore the constant multiplier β in the demand density function in the proof.

To show joint convexity, we need the positive definiteness of the Hessian.

Let the Hessian matrix be denoted by $G(Q,T) = (g_{ij});\ i, j = 1,2$.

$$g_{11} = \alpha^3 \int_Q^\infty y^2 e^{-\alpha y T} e dy;\ g_{22} = \alpha^2 T e^{-\alpha QT};\ g_{12} = g_{21} = \alpha^2 Q e^{-\alpha QT};$$

Clearly, the diagonal elements are positive. The determinant of the Hessian is given by the following expression:

$$D(G) = \alpha^5 T e^{-\alpha QT} \int_Q^\infty y^2 e^{-\alpha y T} dy - \alpha^4 (Q e^{-\alpha QT})^2$$

$$\ge \alpha^5 Q^2 T e^{-\alpha QT} \int_Q^\infty e^{-\alpha y T} dy - \alpha^4 (Q e^{-\alpha QT})^2 = 0.$$

Thus the Hessian is positive definite.

Theorem 5.1: The total supply chain cost of the system as a function of (s, T) is given by:

$$J_{SC}(s,T) = \frac{(A_r + A_m)}{T} + H(s + \frac{DT}{2}) + \frac{(P_r + P_m)}{T} \int_{s+DT}^\infty f(y,T) dy$$

This total system cost function will be used to compare the two systems, VMI and RMI, though the values of (s, T) would be different in the two cases. From Lemma 5.1, we have convexity of the cost function. Convexity of the cost function implies that the optimal decisions can be obtained by solving simultaneously the two first order conditions. Again, without any loss of generality, we ignore the constant multiplier β in the demand density function for ease of exposition. So, we have:

$$\frac{dJ_{SC}}{ds} = 0 \Leftrightarrow \; e^{-\alpha T(s+DT)} = \frac{H}{P\alpha}; \qquad \text{(A2)}$$

$$\frac{dJ_{SC}}{dT} = 0 \; \Leftrightarrow H(\frac{D}{2}-1)T^2 + H(1-\frac{1}{\alpha}\ln[\frac{P\alpha}{H}])T - A - \frac{H}{\alpha^2} = 0 \qquad \text{(A3)}$$

To analyze the VMI system, we need only set $A = A_m$, $H = \rho$ and $P = P_m$ in (A2) and (A3). We will call the resulting optimal values as (s^V, T^V) and $J_{SC}(s^V, T^V)$ as J_{SC}^V. The corresponding optimal values (s^R, T^R) for the RMI system are obtained by setting $A = A_r$ and $P = P_r$ in (A2) and (A3). We let $J_{SC}^R = J_{SC}(s^R, T^R)$.

Choose $\rho = HP_m / P_r$. This choice of ρ implies that $(s_V + DT_V)T_V = (s_R + DT_R)T_R$.

Writing $H(s+\frac{DT}{2}) = \frac{H}{T}[(s+DT)T] - \frac{HD}{2}T$, we have:

$$J_{SC}^V - J_{SC}^R = [(A_r + A_m) + (P_r + P_m)e^{-\alpha T^V(s^V + DT^V)}](\frac{1}{T^V} - \frac{1}{T^R}) + H[(s^V + DT^V)T^V](\frac{1}{T^V} - \frac{1}{T^R}) + \frac{HD}{2}(T^R - T^V)$$

Now notice that solving (A4) yields

$$T = \frac{(\frac{1}{\alpha}\ln[\frac{P\alpha}{H}] - 1) + \sqrt{(1-\frac{1}{\alpha}\ln[\frac{P\alpha}{H}])^2 + 4(\frac{D}{2}-1)(\frac{A}{H} + \frac{1}{\alpha^2})}}{2(\frac{D}{2}-1)}$$

Furthermore, by setting $\rho = HP_m/P_r$ and comparing T^V and T^R using the above equation, we obtain $T^V > T^R$ if $A_m/A_r > P_m/P_r$. This immediately implies that $J_{SC}^V - J_{SC}^R \le 0$. □