A. Proof of Proposition 7

From Equation (3), we can write the profit functions of Firm 1 and Firm 2, before cooperations, as follows

$$\pi_1 = \lambda q_1 \sum_{i\in N_1}\Big(\alpha_i + \sum_{k\notin N_1}\alpha_k\delta_{ki}\Big)(s_i - c_i)$$
$$\pi_2 = \lambda q_2 \sum_{i\in N_2}\Big(\alpha_i + \sum_{k\notin N_2}\alpha_k\delta_{ki}\Big)(s_i - c_i)$$

The profit of Firm 1, after cooperation is given by

$$\pi_1^{\{1,2\}} = \lambda q_1 \sum_{i\in N_1}(s_i - c_i)\Big(\alpha_i + \sum_{\ell\notin N_1\cup N_2}\alpha_\ell\delta_{\ell i}\Big) + \lambda q_1 \sum_{i\in N_2\setminus N_1}(s_i - \beta_{12}s_i)\Big(\alpha_i + \sum_{\ell\notin N_1\cup N_2}\alpha_\ell\delta_{\ell i}\Big)$$
$$+\lambda q_2 \sum_{i\in N_1\setminus N_2}(\beta_{21}s_i - c_i)\Big(\alpha_i + \sum_{\ell\notin N_1\cup N_2}\alpha_\ell\delta_{\ell i}\Big)$$

which can also be written as

$$\Delta\pi_1^{\{1,2\}} = -\lambda q_1 \sum_{i\in N_1}\sum_{\ell\in N_2\setminus N_1}(s_i - c_i)\alpha_\ell\delta_{\ell i} + \lambda q_1 \sum_{i\in N_2\setminus N_1} s_i\Big(\alpha_i + \sum_{\ell\notin N_1\cup N_2}\alpha_\ell\delta_{\ell i}\Big) - \lambda q_2 \sum_{i\in N_1\setminus N_2} c_i\Big(\alpha_i + \sum_{\ell\notin N_1\cup N_2}\alpha_\ell\delta_{\ell i}\Big)$$
$$-\beta_{12}\lambda q_1 \sum_{i\in N_2\setminus N_1} s_i\Big(\alpha_i + \sum_{\ell\notin N_1\cup N_2}\alpha_\ell\delta_{\ell i}\Big) + \beta_{21}\lambda q_2 \sum_{i\in N_1\setminus N_2} s_i\Big(\alpha_i + \sum_{\ell\notin N_1\cup N_2}\alpha_\ell\delta_{\ell i}\Big)$$

Similarly, using Equation (7), we can write the profit of Firm 2 after cooperation as

$$\Delta\pi_2^{\{1,2\}} = -\lambda q_2 \sum_{i\in N_2}\sum_{\ell\in N_1\setminus N_2}(s_i - c_i)\alpha_\ell\delta_{\ell i} + \lambda q_2 \sum_{i\in N_1\setminus N_2} s_i\Big(\alpha_i + \sum_{\ell\notin N_1\cup N_2}\alpha_\ell\delta_{\ell i}\Big) - \lambda q_1 \sum_{i\in N_2\setminus N_1} c_i\Big(\alpha_i + \sum_{\ell\notin N_1\cup N_2}\alpha_\ell\delta_{\ell i}\Big)$$
$$-\beta_{21}\lambda q_2 \sum_{i\in N_1\setminus N_2} s_i\Big(\alpha_i + \sum_{\ell\notin N_1\cup N_2}\alpha_\ell\delta_{\ell i}\Big) + \beta_{12}\lambda q_1 \sum_{i\in N_2\setminus N_1} s_i\Big(\alpha_i + \sum_{\ell\notin N_1\cup N_2}\alpha_\ell\delta_{\ell i}\Big)$$

Then the net change in total profits of these two firms is given by

$$\Delta\pi^C = \Delta\pi_1^{\{1,2\}} + \Delta\pi_2^{\{1,2\}}$$
$$= \lambda q_1 \left\{ \sum_{i\in N_2\setminus N_1}(s_i - c_i)\Big(\alpha_i + \alpha_0\delta_{0i}\Big) - \sum_{i\in N_1}\sum_{\ell\in N_2\setminus N_1}(s_i - c_i)\alpha_\ell\delta_{\ell i}\right\}$$
$$+\lambda q_2 \left\{ \sum_{i\in N_1\setminus N_2}(s_i - c_i)\Big(\alpha_i + \alpha_0\delta_{0i}\Big) - \sum_{i\in N_2}\sum_{\ell\in N_1\setminus N_2}(s_i - c_i)\alpha_\ell\delta_{\ell i}\right\}$$

Now, suppose $\Delta\pi_1^{\{1,2\}} + \Delta\pi_2^{\{1,2\}} \geq 0$ and also define the following parameters

$$A_1 = -\lambda q_1 \sum_{i\in N_1}\sum_{\ell\in N_2\setminus N_1}(s_i - c_i)\alpha_\ell\delta_{\ell i} + \lambda q_1 \sum_{i\in N_2\setminus N_1} s_i\Big(\alpha_i + \sum_{\ell\notin N_1\cup N_2}\alpha_\ell\delta_{\ell i}\Big) - \lambda q_2 \sum_{i\in N_1\setminus N_2} c_i\Big(\alpha_i + \sum_{\ell\notin N_1\cup N_2}\alpha_\ell\delta_{\ell i}\Big)$$

$$A_2 = -\lambda q_2 \sum_{i\in N_2}\sum_{\ell\in N_1\setminus N_2}(s_i - c_i)\alpha_\ell\delta_{\ell i} + \lambda q_2 \sum_{i\in N_1\setminus N_2} s_i\Big(\alpha_i + \sum_{\ell\notin N_1\cup N_2}\alpha_\ell\delta_{\ell i}\Big) - \lambda q_1 \sum_{i\in N_2\setminus N_1} c_i\Big(\alpha_i + \sum_{\ell\notin N_1\cup N_2}\alpha_\ell\delta_{\ell i}\Big)$$

$$B = \lambda q_2 \sum_{i\in N_1\setminus N_2} s_i\Big(\alpha_i + \sum_{\ell\notin N_1\cup N_2}\alpha_\ell\delta_{\ell i}\Big)$$

$$C = \lambda q_1 \sum_{i \in N_2 \setminus N_1} s_i \Big(\alpha_i + \sum_{\ell \notin N_1 \cup N_2} \alpha_\ell \delta_{\ell i} \Big)$$

Using these definitions, we can write $\Delta\pi_1^C$ and $\Delta\pi_1^C$ as

$$\Delta\pi_1^C \;= A_1 + B\beta_{21} - C\beta_{12}$$

$$\Delta\pi_2^C \;= A_2 - B\beta_{21} + C\beta_{12}$$

Now let us consider the set of inequalities

$$A_1 + B\beta_{21} - C\beta_{12} \geq 0$$

$$A_2 - B\beta_{21} + C\beta_{12} \geq 0$$

$$1 \geq \beta_{21} \geq 0$$

$$1 \geq \beta_{12} \geq 0.$$

This set of inequalities has a feasible solution if and only if $-\frac{A_2}{C} \leq 1$ and $-\frac{A_1}{B} \leq 1$. Comparing the terms of $A_1 + B$ and $A_1 + A_2$ yields that, since $\Delta\pi^C = A_1 + A_2 \geq 0$, $A_1 + B \geq 0$ and therefore $-\frac{A_1}{B} \leq 1$. Similarly, Comparing the terms of $A_2 + C$ and $A_1 + A_2$ yields $-\frac{A_2}{C} \leq 1$. As a result, the set of inequalities above has always a feasible solution and therefore one can always find discount factors $0 \leq \beta_{12} \leq 1$ and $0 \leq \beta_{21} \leq 1$ such that $\Delta\pi_1^C \geq 0$ and $\Delta\pi_2^C \geq 0$. □

B. Proofs of Propositions in Section 3.1

Preliminary. When $q_1 = q_2 = q$, $\alpha_1 = \alpha_2 = \alpha$, and $p_1 = p_2 = p$, Equation (11) can be written as

$$\Delta\pi^C = \lambda q p \left(\frac{2\alpha^2 + \alpha(3\theta - 2) - \theta}{\alpha - 1} \right) \tag{23}$$

Let $\Phi(\alpha) = \frac{2\alpha^2 + \alpha(3\theta-2)-\theta}{\alpha-1}$. Since $\frac{\partial^2\Phi}{\partial\alpha^2} = \frac{4\theta}{(\alpha-1)^3} \leq 0$ for all possible values of α and θ, $\Phi(\alpha)$ is concave in α. Further $\Phi(0) = \theta \geq 0$ and $\Phi(\frac{1}{2}) = 1 - \theta \geq 0$. Hence, $\Phi(\alpha)$ is nonnegative. As a result, the net change in total profit, $\Delta\pi^C$ in equation (23), is also nonnegative.

B.1. Proof of Proposition 1

In this case, we assume that $\alpha_1 = \alpha_2 = \alpha$. Define $q_1 = q + \Delta q$ and $q_2 = q - \Delta q$, and let $p_1 = p + \Delta p$ and $p_2 = p - \Delta p$. Then the net change in total profit given in Equation (11) can be written as

$$\Delta\pi^C = \lambda pq\frac{2\alpha^2+\alpha(3\theta-2)-\theta}{(\alpha-1)} - \lambda\Delta p\Delta q\frac{2\alpha^2(2\theta-1)+\alpha(2-3\theta)+\theta}{(1-\alpha)}. \tag{24}$$

For $\Delta\pi^C$ to be nonnegative, the following inequality should hold

$$\Delta p\Delta q \le \frac{pq(-2\alpha^2+\alpha(2-3\theta)+\theta)}{2(2\theta-1)\alpha^2+\alpha(2-3\theta)+\theta}. \tag{25}$$

Since $-2\alpha^2-\alpha(3\theta-2)+\theta \ge 0$ for all $\alpha\in[0,\frac{1}{2}]$ and all $\theta\in[0,1]$, and $-2\alpha^2-\alpha(3\theta-2)+\theta \le 2(2\theta-1)\alpha^2+\alpha(2-3\theta)+\theta$, the right hand side of inequality (25) is always in the interval $[0,1]$. Note that when $\Delta p = 0$, i.e., $p_1 = p_2$, or when $\Delta q = 0$, i.e., $q_1 = q_2$, the term $\Delta p\Delta q$ is 0. Therefore the inequality is satisfied and the cooperation is beneficial. Similarly when $\Delta p\Delta q < 0$, i.e., when $(p_1 > p_2,\ q_1 < q_2)$ or when $(p_1 < p_2,\ q_1 > q_2)$, inequality (25) is satisfied. Otherwise, when $\Delta p\Delta q > 0$, i.e., when $(p_1 > p_2,\ q_1 > q_2)$ or when $(p_1 < p_2,\ q_1 < q_2)$ the term $\Delta p\Delta q$ should not exceed the *threshold* on the right hand side of (25) so that cooperation benefits the two firms. □

B.2. Proof of Proposition 2

Let $\alpha_1 = \alpha+\Delta\alpha, \alpha_2 = \alpha-\Delta\alpha$, $p_1 = p+\Delta p$, and $p_2 = p-\Delta p$. Then, using equation (11), the net change in the total profit can be written as

$$\begin{aligned}\Delta\pi^C \;=\; & \lambda pq\frac{2\alpha\left((1-\alpha)^2-\Delta\alpha^2\right)+\theta\left((1-3\alpha)(1-\alpha)+\Delta\alpha^2\right)}{(1-\alpha)^2-\Delta\alpha^2} \\ & +2\Delta p\Delta\alpha q\lambda\left(1+\frac{\theta(1-2\alpha)}{2\alpha}+\frac{(\alpha^2-\Delta\alpha^2)\theta}{(1-\alpha)^2-\Delta\alpha^2}\right)\end{aligned} \tag{26}$$

Notice that the first term of the summation in equation (26) is the same as the right hand side of equation (23) when $\Delta\alpha = 0$. In the preliminary, we showed that this term is nonnegative. Since $\alpha \ge \Delta\alpha$ and $1-\alpha \ge \Delta\alpha$, first term of the summation in the above expression is also nonnegative. Moreover, since $\alpha \le \frac{1}{2}$,

$$\frac{\theta(1-2\alpha)}{2\alpha}+\frac{(\alpha^2-\Delta\alpha^2)\theta}{(1-\alpha)^2-\Delta\alpha^2} \ge 0.$$

Therefore, the second term of the summation in equation (26) is also nonnegative if $\Delta p\Delta\alpha \ge 0$. On the other hand, $\Delta\pi^C$ will still be nonnegative, for $\Delta p\Delta\alpha < 0$, if $\Delta p\Delta\alpha$ satisfies the following criteria.

$$\Delta p\Delta\alpha \le \frac{-2\alpha^2 p\left((1-\alpha)^2-\Delta\alpha^2\right)+2\theta\alpha\left((1-3\alpha)(1-\alpha)+\Delta\alpha^2\right)}{2\alpha[(1-\alpha)^2-\Delta\alpha^2]+\theta(1-2\alpha)[(1-\alpha)^2-\Delta\alpha^2]+2\theta\alpha(\alpha^2-\Delta\alpha^2)} \tag{27}$$

□

B.3. Proof of Proposition 3

Let $\alpha_1 = \alpha + \Delta\alpha, \alpha_2 = \alpha - \Delta\alpha$, $q_1 = q + \Delta q$, and $q_2 = q - \Delta q$. Then the net change in the total profit can be written as

$$\begin{aligned}\Delta\pi^C \; = \; & \lambda pq \frac{2\alpha\left((1-\alpha)^2 - \Delta\alpha^2\right) + \theta\left((1-3\alpha)(1-\alpha) + \Delta\alpha^2\right)}{(1-\alpha)^2 - \Delta\alpha^2} \\ & -2\Delta q \Delta\alpha p \lambda \left(1 + \frac{\theta(1-2\alpha)}{2\alpha} + \frac{(\alpha^2 - \Delta\alpha^2)\theta}{(1-\alpha)^2 - \Delta\alpha^2}\right)\end{aligned} \tag{28}$$

Following our proof of Proposition 2 and using the similarity of equation (26) and equation (28), notice that $\Delta\pi^C$ is nonnegative when $\Delta q\Delta\alpha \le 0$ or when $\Delta q\Delta\alpha > 0$ and

$$\Delta q\Delta\alpha \le \frac{2q\alpha^2\left((1-\alpha)^2 - \Delta\alpha^2\right) + 2\theta\alpha\left((1-3\alpha)(1-\alpha) + \Delta\alpha^2\right)}{2\alpha[(1-\alpha)^2 - \Delta\alpha^2] + \theta(1-2\alpha)[(1-\alpha)^2 - \Delta\alpha^2] + 2\theta\alpha(\alpha^2 - \Delta\alpha^2)}. \tag{29}$$

□

C. Proofs of Propositions in Section 4

C.1. Proof of Proposition 8

We consider m symmetrical single-product firms. Let $\alpha_0 = 1 - \sum_{i=1}^m \alpha_i$ be the market share of products that are not produced by these m firms, representing the outside option. The net change in total profit of these m firms after cooperation can be written by using Equation (10) as

$$\Delta\pi^C = \sum_{i=1}^m \sum_{j=1, j\neq i}^m \lambda p_i q_j \left(\alpha_i + \alpha_0 \frac{\theta\alpha_i}{1-\alpha_0}\right) - \sum_{i=1}^m \sum_{j=1, j\neq i}^m \lambda p_i q_i \left(\alpha_j \frac{\theta\alpha_i}{1-\alpha_j}\right). \tag{30}$$

Let $\alpha_i = \alpha = \frac{1-\alpha_0}{m}$, $q_i = q$, and $p_i = p$, for $i = 1, ..., m$. The above equation then reduces to

$$\Delta\pi^C = \lambda pqm(m-1)\left(\alpha + \frac{\alpha_0\theta\alpha}{1-\alpha_0} - \frac{\theta\alpha^2}{1-\alpha}\right). \tag{31}$$

Substituting $\alpha_0 = 1 - m\alpha$ into Equation (31) yields

$$\Delta\pi^C = \lambda pq(m-1)\frac{\alpha^2 m + \alpha((m+1)\theta - m) - \theta}{(\alpha - 1)}$$

When $\alpha = 0$, $\Delta\pi^C = \lambda pq(m-1)\theta > 0$. Similarly when $\alpha = \frac{1}{m}$, $\Delta\pi^C = \lambda pq(m-1-\theta) > 0$. $\Delta\pi^C$ reaches its unique maximum of $\lambda pqm(m-1)\left((1-\sqrt{\theta})^2 + \theta\right) > 0$ at $\alpha^* = 1 - \sqrt{\theta}$. Therefore $\Delta\pi^C \ge 0$ for all $0 \le \alpha \le \frac{1}{m}$.

□

C.2. Proof of Proposition 9

Since there are $|M| = m$ single-product firms and $|M| = m$ products, we set $\alpha = \frac{1}{m}$ and $q = \frac{1}{m}$. In this market, let us consider an existing cooperation of k firms, where $k < m$. All the products that are not produced by this cooperation are lumped into a single product with $\alpha_0 = 1 - k\alpha = \frac{m-k}{m}$ as the outside option. The net change in total profit can be written by using Equation (10) as

$$\Delta\pi^{C_k} = \lambda pqk(k-1)\left(\alpha + \frac{\alpha_0\theta\alpha}{1-\alpha_0} - \frac{\theta\alpha^2}{1-\alpha}\right). \tag{32}$$

Since the firms are symmetric, the total benefit is naturally shared equally among k firms. Substituting $\alpha = \frac{1}{m}$, $q = \frac{1}{m}$, and $\alpha_0 = \frac{m-k}{m}$ into the above equation yields the net change in the profit of Firm i, where $i \in C_k$, can be computed as

$$\Delta\pi_i^{C_k} = \lambda p\frac{k-1}{m^2}\left(1 + \frac{\theta(m-k)}{k} - \frac{\theta}{m-1}\right). \tag{33}$$

Now let us consider the case of adding one more member to this cooperation. In this case, the net increase in Firm i's profit can be written directly from Equation (35) by replacing k with $k+1$ that yields

$$\Delta\pi_i^{C_{k+1}} = \lambda p\frac{k}{m^2}\left(1 + \frac{\theta(m-k-1)}{k+1} - \frac{\theta}{m-1}\right). \tag{34}$$

$\pi_i^{C_{k+1}} - \pi_i^{C_k}$ can be derived from Equations (35) and (34) as

$$\Delta\pi_i^{C_{k+1}} - \Delta\pi_i^{C_k} = \frac{\lambda p}{m^2}\left(1 - \theta\left(\frac{1}{m-1} - \frac{m}{k(k+1)} + 1\right)\right). \tag{35}$$

Since $k \leq m-1$, $m \geq 2$, and $\theta \leq 1$, the above term is always positive and therefore adding one more member to any cooperation with $k \leq m-1$ firms is always beneficial for each firm. As a result, a cooperation involving all m firms (*grand coalition*) would naturally form in a market with m symmetric single-product firms. □