Appendix A: Proof of Theorem 1

We first consider k even; the result for k odd easily follows. Let u_j denote the number of sequences σ corresponding to Case j that occur in a k-unit cycle, for all j and r ($j = 2, 3, \ldots, 9$, and $r = 1, \ldots, \frac{k}{2}$) and machines M_i, $i = 1, \ldots, m$. As there are $\frac{k}{2}$ sequences for each machine in a k-unit cycle, we have $\frac{mk}{2} = u_2 + u_3 + u_4 + u_5 + u_6 + u_7 + u_8 + u_9$. Let U_j be the collection of indices of machines in Case j. If a machine M_i has s sequences in Case j, then there will be s instances of i in U_j. u_0 (respectively, u_1) represents the number of visits to I (O) during which the robot unloads (loads) two parts.

By adding residence times corresponding to the possible cases for a machine in a dual-gripper robot cell, we get a lower bound for T_r, the aggregate residence time of the robot at all machines for a k-unit cycle:

$$\begin{aligned} T_r \quad \geq \quad & 2(m+1)k\epsilon + (u_0 + u_1 + u_4 + u_5 + u_7 + 2u_8 + u_9)\theta \\ & + \sum_{i \in U_3} p_i + \sum_{i \in U_5} p_i + 2\sum_{i \in U_6} p_i + \sum_{i \in U_7} p_i + 2\sum_{i \in U_9} p_i. \end{aligned}$$

The computation of required movement time is identical to that of Geismar et al. (2006):

$$T_t = (m+2)k\delta + (-u_0 - u_1 + 2u_2 + u_3 + u_4 - u_9)\delta.$$

Hence, we have lower bound $T_r + T_t$, which can be considered as a fixed amount $((m+2)k\delta + 2(m+1)k\epsilon)$ plus an amount that varies according to which case each sequence is assigned. This is represented for machines $M_1, \ldots, M_m$ by the summation term in expression (1). Table 4 lists the minimum time added to the cycle time for each 2-unit sequence that is assigned to a particular case. The last term in expression (1) corresponds to the time added to the per unit cycle time if $u_0 = u_1 = 1$.

The added time for Case 5 or Case 7 is not included in the summation term because $p_i + \theta \geq \min\{2p_i, 2\theta\}$. Similarly, the added times for Cases 3 and 4 are not included because $p_i + \delta \geq \min\{2p_i, 2\delta\}$ and $\delta + \theta \geq \min\{2\delta, 2\theta\}$, respectively.

For k odd, a lower bound can be found by considering the cycle to be the concatenation of $(k-1)/2$ subsequences $(M^{\ell}_{i,2r-1}, \sigma_1, M^{u}_{i,2r-1}, \sigma_2, M^{\ell}_{i,2r}, \sigma_3, M^{u}_{i,2r}, \sigma_4)$, and a subsequence $(M^{\ell}_{i,k}, \sigma_1, M^{u}_{i,k}, \sigma_2)$. Note that a lower bound for robot actions in a subsequence $(M^{\ell}_{i,k}, \sigma_1, M^{u}_{i,k}, \sigma_2)$

Case	Added time
Case 0, $u_0 = 1$	$\theta - \delta$
Case 1, $u_1 = 1$	$\theta - \delta$
Case 2	2δ
Case 3	$p_i + \delta$
Case 4	$\delta + \theta$
Case 5	$p_i + \theta$
Case 6	$2p_i$
Case 7	$p_i + \theta$
Case 8	2θ
Case 9	$2p_i + \theta - \delta$

Table 4: Variable amount added to the cycle time for each case.

is $(m+2)\delta + 2(m+1)\epsilon + \sum_{i=1}^{m} \min\{p_i, \delta, \theta\}$, which is greater than or equal to the right-hand side of (1). Therefore, (1) is a lower bound for the per unit cycle time for k odd, too. ■

Appendix B: Proofs of Lemmas 1 through 4

Proof of Lemma 1:

a) If $\max p_i \geq (m+2)\delta + 2m\epsilon + (m-1)\theta$, then $S_m^2 = p_h + 2\epsilon + \theta$, which is a lower bound on the per unit cycle time of a dual-gripper robot cell (Geismar et al. 2006). The condition $\theta \leq 3\delta + 2\epsilon$ ensures that $p_h + 2\epsilon + \theta \leq p_h + 3\delta + 4\epsilon$, which is a lower bound for the per unit cycle time of a single-gripper robot cell (Dawande et al. 2002).

b) $p_h \geq (m+2)\delta + 2m\epsilon + (m-1)\theta$ and $\theta \geq 3\delta + 2\epsilon$ imply that $\pi_D = p_h + 3\delta + 4\epsilon \leq p_h + 2\epsilon + \theta$, so π_D is optimal over all single-gripper and dual-gripper cycles. ■

Proof of Lemma 2:

a) $\Omega^1 \geq p_h + 3\delta + 4\epsilon = T(\pi_D)$.

b) $T(\pi_D) = p_h + 3\delta + 4\epsilon \leq p_h + 2\epsilon + \theta \leq \Omega^2$.

c) $\Omega^2 \geq p_h + 2\epsilon + \theta = T(\pi_D) - (3\delta + 2\epsilon - \theta)$.

The condition $\theta \leq 3\delta + 2\epsilon$ implies that the fraction

$$T(\pi_D) \leq \frac{p_h + 3\delta + 4\epsilon}{p_h + 2\epsilon + \theta}\Omega^2$$

is maximized by minimizing p_h. Therefore

$$T(\pi_D) \leq \frac{2(m+1)(\delta+\epsilon)}{(2m-1)\delta+2m\epsilon+\theta}\Omega^2 \leq \frac{2(m+1)(\delta+\epsilon)}{2m(\delta+\epsilon)}\Omega^2 = \frac{m+1}{m}\Omega^2.$$

d) Either S_m^2 is optimal (by Lemma 1) or $T(S_m^2) = (m+2)\delta + 2(m+1)\epsilon + m\theta \leq 2(m+1)(\delta+\epsilon) \leq p_h + 3\delta + 4\epsilon \leq \Omega^1$. Therefore

$$T(S_m^2) \leq \frac{2(m+1)(\delta+\epsilon)}{(2m-1)\delta+2m\epsilon+\theta}\Omega^2 \leq \frac{2(m+1)(\delta+\epsilon)}{(2m-1)\delta+2m\epsilon}\Omega^2 \leq \frac{2m+2}{2m-1}\Omega^2.$$

■

Proof of Lemma 3: If $\theta \leq \delta \leq p_i, \forall i$, then either S_m^2 is optimal (by Lemma 1) or $T(S_m^2) = (m+2)\delta + 2(m+1)\epsilon + m\theta \leq 2(m+1)(\delta+\epsilon) \leq \Omega^1$. That $\theta \leq \delta \leq p_i, \forall i$, implies $T(S_m^2) = \Omega^2$ is proven in Geismar et al. (2006). If $\delta \leq \theta$, and $p_h \leq (2m-1)\delta + 2(m-1)\epsilon$, then $T(\pi_D) = 2(m+1)(\delta+\epsilon) \leq (m+1)(\delta+2\epsilon+\theta) \leq \Omega^2$. That $p_i \geq \delta, \forall i$, implies $T(\pi_D) = \Omega^1$ is proven in Dawande et al. (2002). ■

Proof of Lemma 4: Geismar et al. (2006) show that S_o^2 is optimal over all dual-gripper cycles for $p_i \leq (\delta+\theta)/2, \forall i$. Dawande et al. (2002) show that π_U is optimal over all single-gripper cycles for $p_i \leq \delta, \forall i$. Recall that S_o^2 is a 2-unit cycle. If $\theta \leq \delta$, then

$$\frac{T(S_o^2)}{2} = \frac{(m+2)}{2}\delta + 2(m+1)\epsilon + \frac{(m+2)}{2}\theta + \sum_{i=1}^{m} p_i \leq (m+2)\delta + 2(m+1)\epsilon + \sum_{i=1}^{m} p_i \quad = \quad \Omega^1$$

If $\delta \leq \theta$, then

$$T(\pi_U) = (m+2)\delta + 2(m+1)\epsilon + \sum_{i=1}^{m} p_i \leq \frac{(m+2)}{2}\delta + 2(m+1)\epsilon + \frac{(m+2)}{2}\theta + \sum_{i=1}^{m} p_i \quad = \quad \Omega^2$$ ■

Appendix C: Proof of Theorem 2

Proof of Theorem 2:

Step 1: If $p_i \leq \delta$, $\forall i$, then π_U is optimal by Lemma 4.

Step 2: If $\max_{1 \leq i \leq m} p_i + 3\delta + 4\epsilon \geq 2(m+1)(\delta+\epsilon)$, then π_D is optimal by Lemma 2

Step 3: If $p_i \geq \delta$, $\forall i$, then π_D is optimal by Lemma 3.

Step 4: If $|D_\delta| \geq \frac{5m-4}{9}$, then, from (3), we have

$$\begin{aligned} \Omega^1 &\geq \left[m + \frac{5m-4}{9} + 2\right]\delta + 2(m+1)\epsilon + \sum_{i \in D_\delta^c} p_i \\ &\geq \left(\frac{14}{9}m + \frac{14}{9}\right)\delta + 2(m+1)\epsilon. \end{aligned}$$

Thus,

$$T(\pi_D) = 2(m+1)(\delta+\epsilon) \leq \frac{2(m+1)(\delta+\epsilon)}{\left(\frac{14}{9}m + \frac{14}{9}\right)\delta + 2(m+1)\epsilon}\Omega^1 \leq \frac{9}{7}\Omega^1.$$

We now show tightness. Suppose $m = 8$, $p_1 = p_3 = p_5 = p_7 = 2\delta$, and $p_2 = p_4 = p_6 = p_8 = \nu < \delta$. An optimal cycle is the basic cycle based on the initial partition: $\pi_1 = (A_0, A_7, A_8, A_5, A_6, A_3, A_4, A_1, A_2)$, $T(\pi_1) = LB_1^1 = 14\delta + 18\epsilon + 4\nu$. Algorithm CCell2 outputs π_D, and $T(\pi_D) = 18(\delta+\epsilon)$. Therefore, $T(\pi_D)/T(\pi_1) \to 9/7$ as $\epsilon \to 0$ and $\nu \to 0$.

Step 5: The structure of V_2 and that $|D_\delta| \leq \frac{m+2}{6}$ imply that

$$\begin{aligned} |V_2| &\leq 3|D_\delta| \leq \frac{m+2}{2}, \quad \text{so} \\ \alpha &\leq \frac{\left(m + \frac{m+2}{2} + 2\right)\delta + 2(m+1)\epsilon + \sum_{i \in V_1} p_i}{(m + |D_\delta| + 2)\delta + 2(m+1)\epsilon + \sum_{i \in D_\delta^c} p_i}\Omega^1. \\ &\leq \frac{\frac{3}{2}(m+2)}{\frac{7}{6}(m+2)}\Omega^1 = \frac{9}{7}\Omega^1. \end{aligned}$$

We now investigate the value of $\beta_i, i \in V_2$:

1. By construction, for $i \in D_\delta$, $\beta_i = p_i + 3\delta + 4\epsilon$. By (4), if $T(\widetilde{\pi}_B) = p_i + 3\delta + 4\epsilon$ for some i, then $\widetilde{\pi}_B$ is optimal.

2. For $i \in V_2 \setminus D_\delta$, $\beta_i = p_i + 3\delta + 4\epsilon + \sum_{j \in X_i \cup Y_i}(p_j + \delta + 2\epsilon)$. Since $i \in D_\delta^c$, $X_i \subset D_\delta^c$, and $Y_i \subset D_\delta^c$, we know that $p_i + \sum_{j \in X_i \cup Y_i} p_j \leq \sum_{j \in D_\delta^c} p_j$. Cycle $\widetilde{\pi}_B$ was designed so that $|V_2| \geq 2$. This implies that $|X_i \cup Y_i| \leq m - 2$. Hence,

$$\begin{aligned} \beta_i &\leq \sum_{j \in D_\delta^c} p_j + 3\delta + 4\epsilon + (m-2)(\delta + 2\epsilon) \\ &= \sum_{j \in D_\delta^c} p_j + (m+1)\delta + 2m\epsilon, \; i \in V_2 \setminus D_\delta. \end{aligned}$$

This value is strictly less than LB_1^1. Hence, $\beta_i, i \in V_2 \setminus D_\delta$, will not dominate the cycle time expression.

Step 6: We show that $\alpha/LB_1^1 \leq \frac{9}{7}$ by first proving that

$$|V_2 \setminus D_\delta| \leq \frac{2}{7}(m + |D_\delta| + 2), \tag{6}$$

i.e., by establishing a bound on the number of elements of D_δ^c that the algorithm places into V_2. The largest value for $|V_2 \setminus D_\delta|$ is obtained by maximizing the number of elements $i \in D_\delta$ for which $|Y_i| > \frac{9}{56}(m + |D_\delta| + 2) + 1$. For each such i, two elements of Y_i (the first and the last) will be added to $V_2 \setminus D_\delta$. For example, consider the cell in Figure 5: $m = 12$, $D_\delta = \{1, 6, 11, 12\}$, so $|D_\delta| = 4$ and $\frac{9}{56}(m + |D_\delta| + 2) + 1 = 3.89$. $|Y_1| = |Y_6| = 4$, so indices 2, 5, 7, and 10 will be added to V_2. In general, such a maximal $|V_2 \setminus D_\delta|$ is

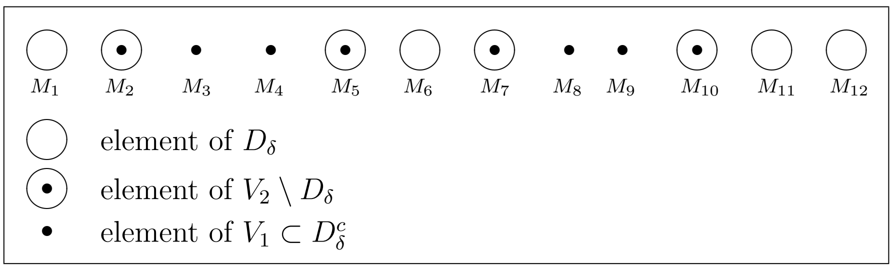

Figure 5: Example of cell with maximal $|V_2 \setminus D_\delta|$.

formed in an m-machine cell if

$$\begin{aligned} D_\delta \;=\; \Bigg\{ 1,\; & 2 + \left\lfloor \frac{9}{56}(m + |D_\delta| + 2) + 2 \right\rfloor, \\ & 3 + 2\left\lfloor \frac{9}{56}(m + |D_\delta| + 2) + 2 \right\rfloor, \\ & \vdots \\ & z + (z-1)\left\lfloor \frac{9}{56}(m + |D_\delta| + 2) + 2 \right\rfloor, \\ & z + (z-1)\left\lfloor \frac{9}{56}(m + |D_\delta| + 2) + 2 \right\rfloor + 1, \ldots, m \Bigg\}. \end{aligned}$$

Note that this uniquely defines the integer z as

$$z = \left\lceil \frac{m}{\lfloor \frac{9}{56}(m + |D_\delta| + 2) + 3 \rfloor} \right\rceil$$

and the size of D_δ as

$$|D_\delta| = m - (z-1)\left\lfloor \frac{9}{56}(m + |D_\delta| + 2) + 2 \right\rfloor.$$

($z = 3$ and $|D_\delta| = 4$ for the example in Figure 5.) In such a cell, $|V_2 \setminus D_\delta| = 2(z-1)$, where $|Y_{i_j}| = \lfloor \frac{9}{56}(m + |D_\delta| + 2) + 2 \rfloor$ for $i_j \in D_\delta$, $j = 1, \ldots, z-1$, and $|Y_{i_j}| = 0$, for $i_j \in D_\delta$, $j = z, \ldots, |D_\delta|$, and $|X_{i_1}| = 0$. Thus, there are $z-1$ intervals in which the first machine is an element of D_δ and the next $\lfloor \frac{9}{56}(m + |D_\delta| + 2) + 2 \rfloor$ machines are elements of D_δ^c. (In Figure 5, M_1 and M_6 are elements of D_δ that are each followed by four elements of D_δ^c.) The final $|D_\delta| - z + 1$ machines (M_{11}, M_{12}) are elements of D_δ. Hence, this requires that there are at least $(z-1)\lfloor (\frac{9}{56}(m + |D_\delta| + 2) + 3) \rfloor + |D_\delta| - z + 1 = (z-1)\lfloor (\frac{9}{56}(m + |D_\delta| + 2) + 2) \rfloor + |D_\delta|$ machines in the cell. To prove (6) we must show that our algorithm cannot add another machine to $V_2 \setminus D_\delta$, if $2(z-1) = \frac{2}{7}(m + |D_\delta| + 2)$. For the algorithm to add another machine to $V_2 \setminus D_\delta$, there must be an additional $\lfloor \frac{9}{56}(m + |D_\delta| + 2) + 1 \rfloor$ elements of D_δ^c between any pair of the last $|D_\delta| - z + 1$ elements of D_δ. We claim that such a configuration is infeasible. If it were feasible, then

$$\begin{aligned} (z-1)\left\lfloor \left(\frac{9}{56}(m + |D_\delta| + 2) + 2 \right) \right\rfloor + |D_\delta| + \left\lfloor \frac{9}{56}(m + |D_\delta| + 2) + 1 \right\rfloor &\leq m \\ \Rightarrow (z-1)\left(\frac{9}{56}(m + |D_\delta| + 2) + 1 \right) + |D_\delta| + \frac{9}{56}(m + |D_\delta| + 2) &\leq m. \qquad (7) \end{aligned}$$

Because $|D_\delta| > \frac{m+2}{6}$, we have $m + |D_\delta| + 2 > \frac{7}{6}(m+2)$ and
$z - 1 = \frac{1}{7}(m + |D_\delta| + 2) > \frac{m+2}{6}$. Thus, (7) must satisfy

$$\begin{aligned} \frac{m+2}{6}\left(\frac{9}{56} \cdot \frac{7}{6}(m+2) + 1 \right) + \frac{m+2}{6} + \frac{9}{56} \cdot \frac{7}{6}(m+2) &\leq m \\ (m+2)(3m+22) + 34(m+2) &\leq 96m \\ 3m^2 - 34m + 112 &\leq 0, \end{aligned}$$

which is a contradiction. Therefore, $|V_2 \setminus D_\delta| \leq \frac{2}{7}(m + |D_\delta| + 2)$.

Thus,

$$\frac{\alpha}{LB_1^1} \leq \frac{(m+|D_\delta|+\frac{2}{7}(m+|D_\delta|+2)+2)\delta+2(m+1)\epsilon+\sum_{j\in V_1} p_j}{(m+|D_\delta|+2)\delta+2(m+1)\epsilon+\sum_{j\in D_\delta^c} p_j}$$

$$\leq \frac{\frac{9}{7}(m+|D_\delta|+2)}{m+|D_\delta|+2} = \frac{9}{7}.$$

We now show that $\beta_i \leq \frac{9}{7}\max\{LB_1^1, LB_2^1\}$, for all $i \in V_2$. First, either $\beta_i \leq \frac{9}{7}LB_2^1$ or

$$\frac{\beta_i}{LB_2^1} = \frac{p_i+3\delta+4\epsilon+\sum_{j\in X_i\cup Y_i}(p_j+\delta+2\epsilon)}{p_i+3\delta+4\epsilon} > \frac{9}{7}$$

$$\Leftrightarrow 7\sum_{j\in X_i\cup Y_i}(p_j+\delta+2\epsilon) > 2(p_i+3\delta+4\epsilon).$$

Therefore,

$$\frac{\beta_i}{LB_1^1} \leq \frac{\frac{9}{2}\sum_{j\in X_i\cup Y_i}(p_j+\delta+2\epsilon)}{(m+|D_\delta|+2)\delta+2(m+1)\epsilon+\sum_{i\in D_\delta^c} p_i}$$

$$= \frac{\sum_{j\in X_i\cup Y_i}(\frac{7}{2}p_j+\frac{9}{2}(\delta+2\epsilon))+\sum_{j\in X_i\cup Y_i} p_j}{(m+|D_\delta|+2)\delta+2(m+1)\epsilon+\sum_{i\in D_\delta^c} p_i}$$

$$\leq \frac{\sum_{j\in X_i\cup Y_i}\left(\frac{7}{2}p_j+\frac{9}{2}(\delta+2\epsilon)\right)}{(m+|D_\delta|+2)\delta+2(m+1)\epsilon} \qquad [X_i\cup Y_i \subset D_\delta^c]$$

$$< \frac{|X_i\cup Y_i|(8\delta+9\epsilon)}{(m+|D_\delta|+2)\delta+2(m+1)\epsilon}. \qquad [p_j<\delta,\ j\in X_i\cup Y_i]$$

Because $|X_i\cup Y_i| \leq \frac{9}{56}(m+|D_\delta|+2)$, for $i \in D_\delta$, and $|D_\delta| < \frac{5m-4}{9}$ (by Step 4),

$$\frac{\beta_i}{LB_1^1} \leq \frac{(m+|D_\delta|+2)(8\cdot\frac{9}{56}\delta+9\cdot\frac{9}{56}\epsilon)}{(m+|D_\delta|+2)\delta+2(m+1)\epsilon} \leq \frac{9}{7}. \qquad \blacksquare$$

Appendix D: Proof of Lemma 5

Proof of Lemma 5: We find the worst case bound and show that it is less than or equal to 3/2. The coefficient of θ in $T(S_a)$ (equation (5)) cannot be greater than its coefficient in $2LB_1^2$, so the largest value for $T(S_a)/2LB_1^2$ occurs when $\theta = 0$. Since $D_2^c = U_5 \cup U_6 \cup U_7 \cup U_9$ in cycle

S_a, we have

$$\begin{aligned} \frac{T(S_a)}{2LB_1^2} &\le \frac{(2m+4-u_0-u_1-u_9)\delta + 4(m+1)\epsilon + \sum_{i\in U_5\cup U_7} p_i + 2\sum_{i\in U_6\cup U_9} p_i}{[2(m+1)-|D_2^c|]\delta + 4(m+1)\epsilon + 2\sum_{i\in D_2^c} p_i} \\ &\le \frac{2m+4-u_0-u_1-u_9}{2(m+1)-|D_2^c|}. \end{aligned} \tag{8}$$

First consider the case in which $U_9 = \emptyset$. The following cell has the largest $|D_2^c|$ for which algorithm DGR-Cell assigns no machine to U_9: $m = 3k$, $p_{3j-2} < (\delta+\theta)/2$, $p_{3j} < (\delta+\theta)/2$, $p_{3j-1} \ge (\delta+\theta)/2$ (but the p_{3j-1} are not large enough to cause positive partial waiting), $j = 1,\ldots,k$, so $|D_2| = k$ and $|D_2^c| = 2k$. Thus, this cell has $k-1$ runs with $s=2$, plus $\{1,m\} \subset D_2^c$, so the algorithm makes the assignments $58668668\cdots86687$. Thus, bounding inequality (8) becomes

$$\frac{T(S_a)}{2LB_1^2} \le \frac{2m+2}{2(m+1)-|D_2^c|} \le \frac{6k+2}{4k+2} < \frac{3}{2}.$$

Now suppose $U_9 \ne \emptyset$. If a run with length $s \ge 2$ begins at $r_1 = 1$ or ends at $r_G + s_G - 1 = m$, then $s-1$ elements (i.e., at minimum one-half of the elements) of that run are placed into U_9. For any other run with length $s \ge 3$, $s-2$ elements (i.e., at minimum one-third of the elements) of that run are placed into U_9. Therefore, because we want to maximize $|D_2^c| - u_9$, we consider only those runs with $s = 3$ and $r_1 \ne 1$, $r_G + s_G - 1 \ne m$. In each of these G runs, for $g = 1,\ldots,G$, we have $r_g \in U_7$, $r_{g+1} \in U_9$, $r_{g+2} \in U_5$; $r_{g+3} \in U_8$ to separate the runs. Hence, the cell that maximizes $|D_2^c| - u_9$ and therefore has the largest value for (8) causes algorithm DGR-Cell to make case assignments so that the cycle begins with one trio assigned 586 and $\ell-1$ trios assigned 686, has a machine in Case 8 (to separate the runs), then has G quartets assigned 7958, and concludes with $j-\ell-1$ trios assigned 686 and one trio assigned 687. Therefore, $m = 4G+3j+1$, $|D_2^c| = 3G+2j$, $u_9 = G$, so

$$\frac{T(S_a)}{2LB_1^2} \le \frac{2(4G+3j+1)+4-G}{2(4G+3j+2)-3G-2j} = \frac{7G+6j+6}{5G+4j+4} \le \frac{3}{2}.$$

■

Appendix E: Proof of Lemma 7

Proof of Lemma 7: We use the fact that if $q \in U_8$, then M_q is reloaded immediately after it is unloaded. We can write the waiting time expressions for any $q \in U_8$ as follows:

$$w_q^1 = \max\left\{0, \widehat{w_q^1} - \sum_{\substack{j=q+1 \\ j \in U_8}}^{m} w_j^2 - \sum_{\substack{j=1 \\ j \in U_8}}^{q-1} w_j^1\right\} \quad (9)$$

$$w_q^2 = \max\left\{0, \widehat{w_q^2} - \sum_{\substack{j=q+1 \\ j \in U_8}}^{m} w_j^1 - \sum_{\substack{j=1 \\ j \in U_8}}^{q-1} w_j^2\right\}, \quad (10)$$

where $\widehat{w_q^1}$ and $\widehat{w_q^2}$ are expressions of the form

$$\widehat{w_q^1} = p_q - a\delta - b\epsilon - c\theta - \sum_{\substack{j=q+1 \\ j \in U_5}}^{m} p_j - \sum_{\substack{j=1 \\ j \in U_6}}^{m} p_j - \sum_{\substack{j=q+1 \\ j \in U_7}}^{m} p_j - 2\sum_{\substack{j=q+1 \\ j \in U_9}}^{m} p_j$$

$$\widehat{w_q^2} = p_q - d\delta - e\epsilon - f\theta - \sum_{\substack{j=1 \\ j \in U_5}}^{q-1} p_j - \sum_{\substack{j=1 \\ j \in U_6}}^{m} p_j - \sum_{\substack{j=1 \\ j \in U_7}}^{q-1} p_j - 2\sum_{\substack{j=1 \\ j \in U_9}}^{q-1} p_j,$$

and $a, b, c, d, e, f \geq 0$ are constants. From (9) and (10) we get

$$\sum_{\substack{j=q+1 \\ j \in U_8}}^{m} w_j^2 + \sum_{\substack{j=1 \\ j \in U_8}}^{q} w_j^1 = \max\left\{\sum_{\substack{j=q+1 \\ j \in U_8}}^{m} w_j^2 + \sum_{\substack{j=1 \\ j \in U_8}}^{q-1} w_j^1,\ \widehat{w_q^1}\right\}$$

$$\sum_{\substack{j=q+1 \\ j \in U_8}}^{m} w_j^1 + \sum_{\substack{j=1 \\ j \in U_8}}^{q} w_j^2 = \max\left\{\sum_{\substack{j=q+1 \\ j \in U_8}}^{m} w_j^1 + \sum_{\substack{j=1 \\ j \in U_8}}^{q-1} w_j^2,\ \widehat{w_q^2}\right\}.$$

Therefore, if $w_q^i > 0$, then $\widehat{w_q^i}$ is the robot's total partial waiting time while M_q^i is processing, $i = 1, 2$. These two equations also imply the following system of $2u_8$ inequalities:

$$\sum_{\substack{j=q+1 \\ j \in U_8}}^{m} w_j^2 + \sum_{\substack{j=1 \\ j \in U_8}}^{q} w_j^1 \geq \max\left\{0, \widehat{w_q^1}\right\}, \quad q \in U_8, \quad (11)$$

$$\sum_{\substack{j=q+1 \\ j \in U_8}}^{m} w_j^1 + \sum_{\substack{j=1 \\ j \in U_8}}^{q} w_j^2 \geq \max\left\{0, \widehat{w_q^2}\right\}, \quad q \in U_8. \quad (12)$$

Let $q' = \text{argmax}_{q \in U_8} \left\{ \max\{0, \widehat{w_q^1}\} + \max\{0, \widehat{w_q^2}\} \right\}$. We show that $W = w_{q'}^1 + w_{q'}^2$, where $w_{q'}^1 = \max\{0, \widehat{w_{q'}^1}\}$, $w_{q'}^2 = \max\{0, \widehat{w_{q'}^2}\}$, and $w_q^1 = w_q^2 = 0$ for $q \neq q'$, is a minimal solution to the system of inequalities (11) and (12). To prove this, we show that (a) $W \geq w_{q'}^1 + w_{q'}^2$, and that (b) $W = w_{q'}^1 + w_{q'}^2$ is a feasible solution to the system of inequalities (11) and (12):

a) That $W \geq w_{q'}^1 + w_{q'}^2$ follows from summing inequalities (11) and (12) for $q = q'$:

$$W = \sum_{\substack{j = q'+1 \\ j \in U_8}}^{m} w_j^2 + \sum_{\substack{j = 1 \\ j \in U_8}}^{q'} w_j^1 + \sum_{\substack{j = q'+1 \\ j \in U_8}}^{m} w_j^1 + \sum_{\substack{j = 1 \\ j \in U_8}}^{q'} w_j^2 \geq \max\left\{0, \widehat{w_{q'}^1}\right\} + \max\left\{0, \widehat{w_{q'}^2}\right\}.$$

b) Suppose $q > q'$. For (11) and (12) to be satisfied, we must have $w_{q'}^1 \geq \widehat{w_q^1}$ and $w_{q'}^2 \geq \widehat{w_q^2}$, which follow from (9) and (10):

$$\begin{aligned} 0 = \quad w_q^1 &= \max\left\{0, \widehat{w_q^1} - w_{q'}^1\right\} \Rightarrow w_{q'}^1 \geq \widehat{w_q^1} \\ 0 = \quad w_q^2 &= \max\left\{0, \widehat{w_q^2} - w_{q'}^2\right\} \Rightarrow w_{q'}^2 \geq \widehat{w_q^2}. \end{aligned}$$

If $q < q'$, then we must have $w_{q'}^2 \geq \widehat{w_q^1}$ and $w_{q'}^1 \geq \widehat{w_q^2}$, which follow similarly from (9) and (10).

By Lemma 6, if $w_{q'}^1 > 0$ and $w_{q'}^2 > 0$, then S_a is optimal. Otherwise, $W = w_{q'}^i > 0$ and $w_{q'}^{3-i} = 0$, $i \in \{1, 2\}$. ∎

Appendix F: Proof of Theorem 3

Proof of Theorem 3: If $\max_{1 \leq i \leq m} p_i \geq (m+2)\delta + 2m\epsilon + (m-1)\theta$, then cycle S_m^2 is optimal by Lemma 1. If $p_i \geq \delta, \forall i$, then cycle S_m^2 is optimal by Lemma 3. If $p_i \leq (\delta + \theta)/2$, $\forall i$, then cycle S_o^2 is optimal by Lemma 4. If $W = 0$ in cycle S_a, then S_a provides a 3/2-approximation to the optimal per unit cycle time by Lemma 5.

We now analyze the case in which $W > 0$ in cycle S_a. The proof of Lemma 7 shows that we need only to consider $W = w_q^i$, $q \in U_8$, where either $i = 1$ or $i = 2$. First suppose that $w_q^1 > 0$. In this case, we can find $T(S_a)$ by adding p_q and the times for robot actions and full waiting that occur between the start of the unloading of M_q^1 and the completion of the loading of this usage.

We first compute the robot movement times in this expression for $T(S_a)$. After unloading M_q^1, rotating its grippers, and loading M_q^2, the robot travels to each machine M_j, $j > q$, $j \in U_5 \cup U_6 \cup U_7 \cup U_8$, then to O (if $u_1 = 0$), to I, and to each machine $M_1, \ldots, M_q$, before unloading M_q^2, rotating its grippers, and reloading M_q^1. Hence, the total movement time in S_a is $(q + 2 - u_1 + |\{j : j > q, j \in U_5 \cup U_6 \cup U_7 \cup U_8\}|)\delta$.

Now consider the load/unload, gripper rotation, and full waiting times. For clarity, we explicitly calculate only the load/unload times within the text. After unloading M_q^1 but before visiting I, the robot unloads each M_j for which $j \in U_5$ and $j > q$, and it loads each M_j for which $j \in U_7$ and $j > q$ (this requires time $|\{j : j > q, j \in U_5 \cup U_7\}|\epsilon$). During this same segment of the cycle, the robot loads, waits, and unloads (respectively, unloads, rotates grippers, and loads) at each M_j, $j \in U_6$ (resp., $j \in U_8$), $j \geq q$ (time $2|\{j : j \geq q, U_6 \cup U_8\}|\epsilon$). Before visiting I, the robot may load O (time $(1 - u_1)\epsilon$). After visiting I (time $(1 + u_0)\epsilon$) but before loading M_q^1, the robot loads, waits, unloads, rotates grippers, and reloads (respectively unloads, rotates grippers, reloads, waits, and unloads) at each M_i, $i \in U_5$ (resp., $i \in U_7$), $i < q$ (time $3|\{i : i < q, i \in U_5 \cup U_7\}|\epsilon$). At each M_i, $i \in U_6$ (resp., $i \in U_8$), $i \leq q$, the robot loads, waits, and unloads (resp., unloads, rotates grippers, and loads) (time $2|\{i : i \leq q, U_6 \cup U_8\}|\epsilon$). For each machine M_i, $i \in U_9$, $i < q$, the robot loads, waits, unloads, rotates grippers, reloads, waits, and unloads (time $4|\{i : i < q, i \in U_9\}|\epsilon$). This analysis generates the following cycle time expression if $W = w_q^1$:

$$\begin{aligned}
T(S_a) \quad = \quad & p_q + (q + 2 - u_1 + |\{j : j > q, j \in U_5 \cup U_6 \cup U_7 \cup U_8\}|)\delta \\
& + [4|\{i : i < q, i \in U_9\}| + 3|\{i : i < q, i \in U_5 \cup U_7\}| + 2(|\{U_6 \cup U_8\}| + 1) \\
& + |\{j : j > q, j \in U_5 \cup U_7\}| + 2 + u_0 - u_1]\epsilon \\
& + (|U_8| + |\{i : i < q, i \in U_5 \cup U_7 \cup U_9\}| + u_0 + 1)\theta \\
& + 2 \sum_{\substack{i = 1 \\ i \in U_9}}^{q-1} p_i + \sum_{\substack{i = 1 \\ i \in U_5 \cup U_7}}^{q-1} p_i + \sum_{i \in U_6} p_i.
\end{aligned}$$

Note that $q + |\{j : j > q, j \in U_5 \cup U_6 \cup U_7 \cup U_8\}| \leq m$, $u_5 + u_7 + u_8 + u_9 \leq m$, and $0 \leq u_0, u_1 \leq 1$.

Thus,

$$T(S_a) \leq p_q + (m+2)\delta + 4(m+1)\epsilon + (m+2)\theta + 2\sum_{i\notin U_8} p_i. \tag{13}$$

We further bind $T(S_a)$ by binding p_q by using LB_2^2. If $T(S_a)/2LB_2^2 \leq 3/2$, then the theorem is proven. Otherwise, $T(S_a) > 3p_q + 3\theta + 6\epsilon$, so, by (13), we have

$$\begin{aligned}
3p_q + 3\theta + 6\epsilon &< p_q + (m+2)\delta + 4(m+1)\epsilon + (m+2)\theta + 2\sum_{i\notin U_8} p_i \\
p_q &< \frac{m+2}{2}\delta + (2m-1)\epsilon + \frac{m-1}{2}\theta + \sum_{i\notin U_8} p_i \\
\Rightarrow T(S_a) &< \frac{3}{2}(m+2)\delta + (6m+3)\epsilon + \frac{3}{2}(m+1)\theta + 3\sum_{i\notin U_8} p_i.
\end{aligned}$$

Recall that

$$\begin{aligned}
2LB_1^2 &= (2m+2-|D_2^c|)(\delta+\theta) + 4(m+1)\epsilon + 2\sum_{i\in D_2^c} p_i \\
&= (2m+2-u_5-u_6-u_7-u_9)(\delta+\theta) + 4(m+1)\epsilon + 2\sum_{i\notin U_8} p_i.
\end{aligned}$$

Since $u_5+u_6+u_7+u_9 \leq m-1$, it follows that

$$2LB_1^2 \geq (m+3)(\delta+\theta) + 4(m+1)\epsilon + 2\sum_{i\notin U_8} p_i.$$

Therefore,

$$\frac{T(S_a)}{LB_1^2} < \frac{\frac{3}{2}(m+2)\delta + (6m+3)\epsilon + \frac{3}{2}(m+1)\theta + 3\sum_{i\notin U_8} p_i}{(m+3)(\delta+\theta) + 4(m+1)\epsilon + 2\sum_{i\notin U_8} p_i} < \frac{3}{2}.$$

The proof for $w_q^2 > 0$ is similar.

To demonstrate asymptotic tightness, let $p_i = \nu < (\delta+\theta)/2$ for $i = 1, \ldots, m-1$, and $p_m = \frac{m+1.9}{2}\delta + \frac{m-1}{2}\theta$. It follows that

$$\begin{aligned}
T(S_a) &= p_m + (m+2)\delta + (4m+2)\epsilon + (m+2)\theta + (2m-3)\nu \\
&= \left(\frac{3}{2}m + 2.95\right)\delta + (4m+2)\epsilon + \frac{3}{2}(m-1)\theta + (2m-3)\nu, \\
2LB_1^2 &= (m+3)(\delta+\theta) + 4(m+1)\epsilon + 2(m-1)\nu \\
2LB_2^2 &= 2p_m + 4\epsilon + 2\theta = (m+1.9)\delta + 4\epsilon + (m+1)\theta.
\end{aligned}$$

Therefore,

$$\frac{T(S_a)}{2LB_1^2} = \frac{\left(\frac{3}{2}m + 2.95\right)\delta + (4m+2)\epsilon + \frac{3}{2}(m-1)\theta + (2m-3)\nu}{(m+3)(\delta+\theta) + 4(m+1)\epsilon + 2(m-1)\nu} \longrightarrow \frac{3}{2}$$

$$\frac{T(S_a)}{2LB_2^2} = \frac{\left(\frac{3}{2}m + 2.95\right)\delta + (4m+2)\epsilon + \frac{3}{2}(m-1)\theta + (2m-3)\nu}{(m+1.9)\delta + 4\epsilon + (m+1)\theta} \longrightarrow \frac{3}{2}$$

as $m \to \infty$, $\epsilon \to 0$, and $\nu \to 0$. ■